Mathematics: analysis and approaches
Standard level
Paper 2

Name

worked solutions

Date: ____________________

1 hour 30 minutes

SOLUTION KEY

Instructions to candidates

- Write your name in the box above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written in the answer boxes provided.
- Section B: answer all questions on the answer sheets provided. Write your name on each answer sheet and attach them to this examination paper.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches SL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[80 marks]**.

exam: 7 pages

Section A (37 marks)

1. [Maximum mark: 7]

An amount of £1000 is invested on January 6th 2025 in a bank account with an interest rate of 2.5%, compounded annually. There are no further deposits or withdrawals and the annual interest earned is added to the account each year on January 6th.

(a) Find the amount in the account after six years. [3]

(b) Find the number of years after which the account first exceeds £1500. [4]

Solution:

(a) $P = 1000\left(1+\frac{2.5}{100}\right)^6 \approx 1159.6934...$ after 6 years, the account has a balance of £1159.69

(b) $1000\left(1+\frac{2.5}{100}\right)^n > 1500 \Rightarrow 1.025^n > 1.5$; $1.025^n = 1.5 \Rightarrow \log\left(1.025^n\right) = \log(1.5)$

$n\log(1.025) = \log(1.5) \Rightarrow n = \frac{\log(1.5)}{\log(1.025)} \approx 16.42...$ thus, account exceeds £1500 in 17 years

2. [Maximum mark: 6]

Given that θ is an obtuse angle and $\sin\theta = \frac{3}{5}$, find the **exact** value of each of the following expressions:

(a) $\cos\theta$ [2]

(b) $\sin 2\theta$ [2]

(c) $\cos 2\theta$ [2]

Solution:

(a) $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \cos\theta = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}$; θ is obtuse $\Rightarrow \cos\theta = -\frac{4}{5}$

(b) $\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) = -\frac{24}{25}$

(c) $\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$

3. [Maximum mark: 6]

Events A and B are such that $\mathrm{P}(A \cup B) = 0.9$, $\mathrm{P}(A \cap B) = 0.45$ and $\mathrm{P}(A|B) = 0.75$.

(a) Find $\mathrm{P}(B)$. [2]

(b) Find $\mathrm{P}(A)$. [2]

(c) Hence, show that events A and B are independent. [2]

Solution:

(a) $\mathrm{P}(A|B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \quad \Rightarrow \quad 0.75 = \dfrac{0.45}{\mathrm{P}(B)} \quad \Rightarrow \quad \mathrm{P}(B) = \dfrac{0.45}{0.75} = 0.6$

(b) $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) \quad \Rightarrow \quad 0.9 = \mathrm{P}(A) + 0.6 - 0.45 \quad \Rightarrow \quad \mathrm{P}(A) = 0.75$

(c) A and B are independent events because $\mathrm{P}(A) = \mathrm{P}(A|B)$

4. [Maximum mark: 5]

Consider the function $y = p + \dfrac{p^2}{x} + x^2,\ x \neq 0$,where p is a constant.

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. [1]

(b) The graph of the function has a local minimum point at $(2,8)$. Find the value of p. [4]

Solution:

(a) $y = p + p^2x^{-1} + x^2 \quad \Rightarrow \quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = -p^2x^{-2} + 2x = -\dfrac{p^2}{x^2} + 2x$

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{p^2}{x^2} + 2x = 0$ when $x = 2$: $-\dfrac{p^2}{4} + 4 = 0 \quad \Rightarrow \quad p^2 = 16 \quad \Rightarrow \quad p = \pm 4$

$p = 4$: $y = 4 + \dfrac{16}{x} + x^2 \quad \Rightarrow \quad y(2) = 4 + \dfrac{16}{2} + 2^2 = 16$

$p = -4$: $y = -4 + \dfrac{16}{x} + x^2 \quad \Rightarrow \quad y(2) = -4 + \dfrac{16}{2} + 2^2 = 8$

Therefore, $p = -4$

5. [Maximum mark: 6]

A multiple choice test consists of twelve questions. Each question has four answers. Only one of the answers is correct. For each question, Emma randomly chooses one of the fours answers.

(a) Write down the expected number of questions Emma answers correctly. [1]

(b) Find the probability that Emma answers exactly four questions correctly. [2]

(c) Find the probability that Emma answers more than four questions correctly. [3]

Solution:

(a) $\mathrm{E}(X) = 12 \cdot \frac{1}{4} = 3$

(b) $X \sim \mathrm{B}\left(12, \frac{1}{4}\right);\quad \mathrm{P}(X=4) = \binom{12}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^8 = 0.19358... \approx 0.194$

or calculate with GDC: $\mathrm{binomPdf}\left(12, \frac{1}{4}, 4\right)$ 0.193577706814

(c) $\mathrm{P}(X > 4) = 1 - \left[\mathrm{P}(X=0) + \mathrm{P}(X=1) + \mathrm{P}(X=2) + \mathrm{P}(X=3) + \mathrm{P}(X=4)\right]$

$= 1 - [0.031676... + 0.12671... + 0.23229... + 0.25810... + 0.19358...] = 1 - 0.84236... \approx 0.158$

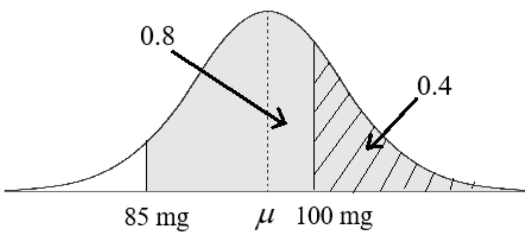
or calculate with GDC: $\mathrm{binomCdf}\left(12, \frac{1}{4}, 5, 12\right)$ 0.157643676073

6. [Maximum mark: 7]

It is known that two out of five cups of coffee served at Bella's Coffee Shop contain more than 100 mg of caffeine. It is also known that four out of five cups served at Bella's contain more 85 mg of caffeine.

Assuming the amount of caffeine in a cup of coffee at Bella's is modelled by a normal distribution, find the mean and standard deviation of the caffeine content in a cup of coffee served at Bella's.

Solution: $\mu = ?\quad \sigma = ?\qquad z = \frac{x-\mu}{\sigma}$



$\mathrm{P}(X < 85) = 1 - \mathrm{P}(X > 85) = 0.2 \quad \Rightarrow \quad z = -0.84162...$

$-0.84162... = \frac{85-\mu}{\sigma} \quad \Rightarrow \quad \mu + (-0.84162...)\sigma = 85$

$\mathrm{P}(X < 100) = 1 - \mathrm{P}(X > 100) = 0.6 \quad \Rightarrow \quad z = 0.25335...$

$0.25335... = \frac{100-\mu}{\sigma} \quad \Rightarrow \quad \mu + (0.25335...)\sigma = 100$

$\begin{cases} \mu + (-0.84162...)\sigma = 85 \\ \mu + (0.25335...)\sigma = 100 \end{cases} \Rightarrow$ use linear equation solver on GDC $\Rightarrow \mu \approx 96.5$ mg, $\sigma = 13.7$ mg

invNorm(0.2,0,1)	-0.841621233465
-0.84162123346456→z1	-0.841621233465
invNorm(0.6,0,1)	0.253347101143
0.25334710114285→z2	0.253347101143

$\mu = m, \sigma = s$

$\mathrm{linSolve}\left(\begin{cases} m + z1 \cdot s = 85 \\ m + z2 \cdot s = 100 \end{cases}, \{m, s\}\right)$

{96.5293914015,13.6990262877}

Section B (43 marks)

7. [Maximum mark: 12]

A company that manufactures car tires conducts an experiment to determine how a certain type of tire maintains its air pressure over time. A new tire is fitted to a wheel. The tire is then inflated to its recommended pressure of 39 psi (pounds per square inch) and the tire is placed in a temperature controlled room. At three-month intervals, the air pressure of the tire is measured giving the following results.

time (x months)	0	3	6	9	12	15	18	21	24
tire pressure (y psi)	39.0	37.2	35.6	34.7	33.5	32.2	30.6	29.2	28.1

(a) Write down the equation of a straight line model for the association between time and tire pressure, i.e. an equation of the regression line of y (pressure) on x (time). [2]

(b) Comment on the strength of the association between time and tire pressure. [2]

(c) Use your straight line (linear regression) model to interpret the meaning of
(i) the gradient
(ii) the y-intercept. [2]

(d) Estimate the air pressure (psi) of the tire 20 months after being fitted to the wheel. [2]

(e) Do not give numerical answers for this question. Comment on the appropriateness of using your model to:
(i) estimate the tire pressure after three years;
(ii) estimate the number of months it would take for the tire pressure to decrease to 30 psi. [4]

Solution:

(a) $y = -0.445x + 38.7$

(b) $r = -0.998...$
There is a very strong negative linear association.

(c) (i) Rate of decrease of tyre pressure per month.

(ii) Initial tyre pressure.

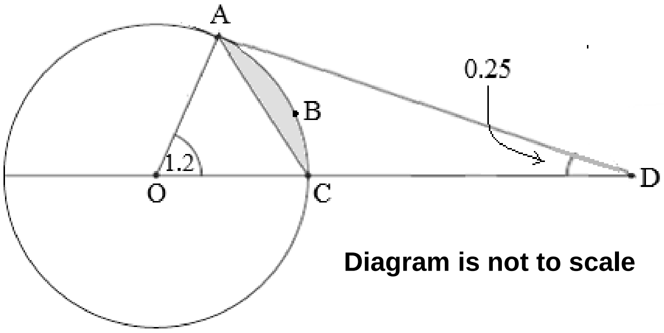
(d) 29.8 psi

(e) (i) This would be extrapolation which would not be an appropriate use of the regression line.

(ii) A regression line cannot predict x from y so this is not appropriate.

8. [Maximum mark: 18]

The diagram below shows a circle with centre O and radius 6 cm.



The points A, B and C lie on the circle. The point D is outside the circle and lies on (OC). Angle $\text{AOC} = 1.2$ radians and angle $\text{ADO} = 0.25$ radians.

(a) Find the area of the sector OABC. [3]

(b) Find the area of the shaded region bounded by the chord AC and the arc ABC. [4]

(c) Find AD. [3]

(d) Find OD [4]

(e) Find the area of the region ABCD. [4]

Solution:

(a) $A = \frac{1}{2}\theta r^2 \quad \Rightarrow \quad \frac{1}{2}(1.2)6^2 = 21.6 \text{ cm}^2$

(b) area of shaded region = area of sector OABC – area of triangle OAC

$$= 21.6 - \frac{1}{2}\cdot 6\cdot 6\cdot \sin(1.2) = 21.6 - 16.7767... \approx 4.82 \text{ cm}^2$$

(c) sine rule: $\frac{\text{OA}}{\sin(\text{ODA})} = \frac{\text{AD}}{\sin(\text{AOC})} \quad \Rightarrow \quad \frac{6}{\sin(0.25)} = \frac{\text{AD}}{\sin(1.2)} \quad \Rightarrow \quad \text{AD} \approx 22.6 \text{ cm}$

(d) $\text{OAD} = \pi - (1.2 + 0.25) \approx 1.69159...$

sine rule: $\frac{\text{OA}}{\sin(\text{ODA})} = \frac{\text{OD}}{\sin(\text{OAD})} \quad \Rightarrow \quad \frac{6}{\sin(0.25)} = \frac{\text{OD}}{\sin(1.69159...)} \quad \Rightarrow \quad \text{OD} \approx 24.1 \text{ cm}$

[**note**: use 'full calculator accuracy' values for AD and OD in calculations for part (e) below]

(e) area region ABCD = area triangle OAD – area sector OABC

area of triangle OAD $= \frac{1}{2}\cdot 6(24.0751...)\sin(1.2) \approx 67.3168...\text{cm}^2$

OR area triangle OAD $= \frac{1}{2}(22.6037...)(24.0751...)\sin(0.25) \approx 67.3168...\text{cm}^2$

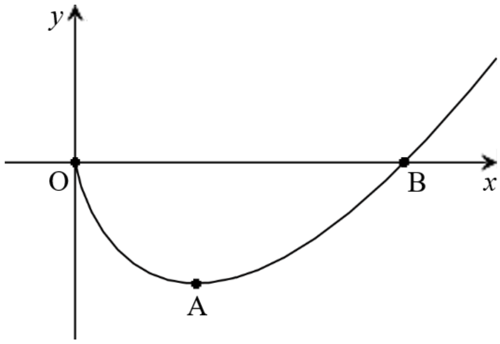
thus, area of region ABCD $= 67.3168... - 21.6 \approx 45.7 \text{ cm}^2$

9. [Maximum mark: 17]

Consider the function $g_n(x)=\begin{cases} x\ln x - nx, & x>0 \\ 0, & x=0 \end{cases}$, where $n=0,1,2,\ldots$

(a) Find the derivative of $g_n(x)$, $x>0$. [2]

The graph of the function $g_n(x)$ is shown. One x-intercept is at the origin O and the other x-intercept is at point B. The graph has an absolute minimum at point A.



(b) Show that the x-coordinate of A is e^{n-1}. [2]

(c) Find the x-coordinate of B. [3]

(d) Find the equation of the tangent to the graph of $g_n(x)$ at B. [3]

(e) Find the area bounded by the graph of $g_n(x)$ and the x-axis when $n=1$. [3]

(f) Show that the x-coordinates of the minimum point on the graph of $g_n(x)$, for consecutive values of n, form a geometric sequence. [4]

Solution:

(a) $g_n(x)=x\ln x-nx,\ x>0$; $\quad g_n'(x)=\left(\ln x + x\cdot\frac{1}{x}\right)-n=1-n+\ln x$

(b) A is a minimum of $g_n(x)$, thus its x-coordinate exists where $g_n'(x)=0$

$g_n'(x)=1-n+\ln x=0 \;\Rightarrow\; \ln x=n-1 \;\Rightarrow\; x=e^{n-1}$ ***Q.E.D.***

(c) x-coordinate of B when $g_n(x)=0$;

$g_n(x)=x\ln x-nx=0 \;\Rightarrow\; x(\ln x-n)=0 \;\Rightarrow\;$ ~~$x=0$~~ or $\ln x=n$

$\ln x=n \;\Rightarrow\; x=e^n$ thus, the x-coordinate of B is e^n

(d) at $x=e^n$, $g_n'(x)=\ln(e^n)-n+1=n-n+1=1$; hence, gradient of tangent at $B(e^n,\,0)$ is 1

equation of tangent at B: $y-0=1\cdot(x-e^n) \;\Rightarrow\; y=x-e^n$

(e) area of bounded region $=\left|\int_0^e (x\ln x-x)\,dx\right| \approx 1.84726...$ area ≈ 1.85 units²

(f) from part (a), the x-coordinate of the minimum point in terms of n is e^{n-1}

hence, the sequence of x-coordinates of the minimum point for $n=0,1,2,3,\ldots$ is $e^{-1}, 1, e, e^2, \ldots$

which is a geometric sequence where $u_1=e^{-1}$ and $r=e$